S.-T. Yau College Student Mathematics Contests 2026

Computational and Applied Mathematics

(6 problems)

Problem 1. Consider the equation

$$\alpha\partial_t u(t,x) + \beta\partial_x u(t,x) - \gamma\partial_{xx} u(t,x) = f(x), \tag{1}$$

for $(t,x) \in (0,T)\times(0,1)$, with $f \in L^2(0,1)$, $T>0$, boundary condition $u(t,0)=u(t,1)=0$ for all $t>0$ and initial condition $u(0,x)=0$. Parameters satisfy $\alpha>0$, $\beta\in\mathbb{R}$ and $\gamma>0$. Let $\mathcal{T}_h$ be a uniform mesh partitioning $(0,1)$, i.e., a collection of intervals $[ih,(i+1)h]$ with $i=0,1,\dots,N$ and $h=1/(N+1)$.

(a). Write the fully discrete variational formulation of (1) using the continuous piecewise linear finite element method ($\mathbb{P}_1$ Lagrange FEM) in space and implicit Euler method in time. Denoting the time step by τ and $t_i = i\tau$ for all $i\in\mathbb{N}$.

(b). Prove the L^2 stability estimate $\|u_h^n\| \le C\|f\|$ with a constant $C>0$ independent of h,τ,n.

(c). Let $\{\phi_i\}_{1\le i\le N}$ be the global Lagrange shape functions associated with the nodes $x_i := ih$ for $i=1,\dots,N$. Denoting by $u_h^i := \sum_{1\le j\le N} U_j^i\phi_j$ the approximation of u at t_i, write the algebraic linear system solved by $(U_1^i,\dots,U_N^i)$.

Problem 2. Let $\Omega\subset\mathbb{R}^d$ be a bounded domain, $d\le 3$. Consider the functional $J : H_0^1(\Omega)\to\mathbb{R}$:

$$J(u) = \int_\Omega \left(\frac{1}{2}|\nabla u(x)|^2 + \frac{1}{4}u(x)^4 - f(x)u(x)\right)dx,$$

where $f\in L^2(\Omega)$.

(a). Compute the Fréchet derivative (gradient) $\nabla J(u)$ and the second Fréchet derivative (Hessian) $\nabla^2 J(u)$. Prove that J is strictly convex.

(b). Write the Newton's method for finding the minimizer u^* of $J(u)$. Assuming the initial guess u^0 is sufficiently close to the solution u^*, prove the second order convergence of the Newton's method: there exists $C>0$ such that $\|u_{k+1}-u^*\|_{H^1} \le C\|u_k - u^*\|_{H^1}^2$.

(c). Let $s_k = u_{k+1}-u_k$ and $y_k = \nabla J(u_{k+1}) - \nabla J(u_k)$. A Quasi-Newton method resorts to an approximate Hessian B_{k+1} satisfying the secant equation: $B_{k+1}s_k = y_k$. A BFGS update of B_{k+1} is

$$B_{k+1}v = B_k v - \frac{\langle B_k s_k, v\rangle}{\langle B_k s_k, s_k\rangle}B_k s_k + \frac{\langle y_k, v\rangle}{\langle y_k, s_k\rangle}y_k,$$

for any test function $v\in H_0^1(\Omega)$. Assuming B_k is positive definite, prove that above BFGS update is well-defined.

Problem 3. Consider the initial value problem over $\mathbb{R}^N$ in the form

$$x' = f(t,x) \qquad x(0) = x_0 \in \mathbb{R}^N,$$

where $f : [0,T]\times\mathbb{R}^N \to \mathbb{R}^N$ is smooth. Consider the family of one-step methods

$$x_{n+1} = x_n + (1-b)hf(t_n,x_n) + bhf(t_{n+1},x_{n+1}), \tag{2}$$

where $h = t_{n+1}-t_n$ for any $n\ge 0$ is a uniform step size and $b\in[0,1]$ is a constant.

(a). Find the value of b so that the local truncation error is $\mathcal{O}(h^3)$.

(b). Apply the method (2) to $x' = \lambda x$, $x(0) = x_0 \in \mathbb{R}$. Find the function $g(\cdot)$ such that $x_n = g(h\lambda)^n x_0$.

(c). Determine the values of b so that this method is A-stable.

Problem 4. Let A be an invertible $N \times N$ matrix. The shifted QR iteration for a given sequence of shifts $\{\sigma_n\}$ is defined by

$$A_0 = A, \quad A_n - \sigma_n I_N = Q_n R_n, \quad A_{n+1} = R_n Q_n + \sigma_n I_N,$$

where I_N is the $N \times N$ identity matrix, Q_n is orthogonal and R_n is upper triangular with positive diagonal entries.

(a). Prove if no σ_n is an eigenvalue of A then the sequences $\{A_n\}$, $\{Q_n\}$ and $\{R_n\}$ are uniquely defined and satisfy

$$A_{n+1} = Q_n^T A_n Q_n, \quad A_{n+1} = R_n A_n (R_n)^{-1}.$$

(b). Suppose A is a symmetric 2×2 matrix with eigenvalues λ_1 and λ_2. Let $\sigma_0 = \lambda_1$. Find A_1.

(c). Let $\hat{Q}_n = Q_0 \dots Q_{n-1}$ and $\hat{R}_n = R_{n-1} \dots R_0$ for $n \geq 1$. Prove $\hat{Q}_{k+1}\hat{R}_{k+1} = \prod_{i=0}^{k}(A - \sigma_i I_N)$.

Problem 5. Let $A \in \mathbb{R}^{n \times n}$ be an n by n real matrix and $\sigma_i(A)$ be its i-th largest singular value. A vector $x \in \mathbb{R}^n$ such that $Ax = x$ is called a fixed point of A.

(a). Assume $\sigma_1(A) \leq 1$, show that every fixed point of A is a fixed point of its transpose A^T.

(b). Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ to verify that the assertion in (a) is not generally true without assuming $\sigma_1(A) \leq 1$.

(c). Assume $AA^T = A^T A$, show that every fixed point of A is a fixed point of its transpose A^T.

Problem 6. Let the subdifferential of a convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at a point x be denoted by $\partial f(x)$. Recall that a vector $x^* \in \mathbb{R}^n$ is a subgradient of f at x if it satisfies:

$$f(z) \geq f(x) + \langle x^*, z - x \rangle, \quad \forall z \in \mathbb{R}^n.$$

Namely, $\partial f(x) = \{x^* : f(z) \geq f(x) + \langle x^*, z - x \rangle, \ \forall z \in \mathbb{R}^n\}$. Here, $\langle x, y \rangle = x^T y$ for any vectors x and y in $\mathbb{R}^n$. We assume $n \geq 2$ (the $n = 1$ case is trivial).

(a). Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(x) = \max_{i=1,\dots,n} x_i, \quad x = (x_1, \dots, x_n)^T \in \mathbb{R}^n.$$

Compute the subdifferential $\partial f(0)$.

(b). Let

$$g(x) = \max_{i=1,\dots,n} x_i + \delta_{\mathbb{R}^n_+}(x),$$

where $\delta_{\mathbb{R}^n_+}$ is the indicator function of the non-negative orthant $\mathbb{R}^n_+$ (i.e., $\delta_{\mathbb{R}^n_+}(x) = 0$ if $x \geq 0$, and $+\infty$ otherwise). Show that

$$\partial g(0) = \partial f(0) - \mathbb{R}^n_+.$$

[Here, one may use the fact that $\partial g(x) = \partial f(x) + \partial \delta_{\mathbb{R}^n_+}(x)$ for $g(x) = f(x) + \delta_{\mathbb{R}^n_+}(x)$.]